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LETTER TO THE EDITOR

Scaling properties of a transformation defined on cellular automaton rules

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Abstract. A one-parameter family of transformations defined on the set of all one-dimensional cellular automata is studied. The class of a given cellular automaton is unchanged under the set of these transformations. For class 3 cellular automata, transformed statistical quantities satisfy simple scaling properties.

Cellular automata (CA) are discrete dynamical systems. They consist of a lattice of sites, each taking on one of the values 0, 1, ..., $k-1$. The values of the sites evolve synchronously in discrete time steps according to a definite rule. In this letter we shall consider one-dimensional CA with $k=2$. The value site i at time t is denoted s_i^t, and the rule f is of the form

$$s_i^t = f(s_{i-r}^{t-1}, s_{i-r+1}^{t-1}, \ldots, s_{i+r}^{t-1})$$

where the parameter r is called the range of the rule. The value of site i at time t is therefore determined by the values of the sites of a neighbourhood containing $2r+1$ sites at time $t-1$.

Based on investigation of a large sample of CA, Wolfram (1984) has shown that probably all CA fall into four qualitative classes. Class 1 CA evolve, from almost all initial states, to a unique homogeneous state in which all sites have the same value. Class 2 CA yield separated simple stable or periodic structures. Class 3 CA exhibit chaotic patterns. The statistical properties of these patterns are typically the same for almost all initial states. In particular, the density of non-zero sites tends to a fixed value. The evolution of class 4 CA leads to complex localised or propagating structures.

In what follows a transformation T_b, characterised by a positive odd integer b, is defined. Numerical simulations show that the CA evolving according to the rule f and the transformed rules $T_b f$ ($b=3, 5, \ldots$) belong to the same class.

In the particular case of class 3 CA, for a given rule f the probability distribution of the asymptotic density of non-zero sites of CA evolving according to rules $T_b f$ ($b=1, 3, 5, \ldots$) exhibits a simple scaling property.

In order to build up T_b, consider the set

$$\{s_{j-(b-1)/2}, s_{j-(b-3)/2}, \ldots, s_{j+(b-1)/2}\}$$

which forms a block of length b (b is odd) centred at j. With a block associate an integral valued variable B_j, called a block variable, such that

$$B_j = \begin{cases} 1 & \text{if } S_{b,j} > b/2 \\ 0 & \text{if } S_{b,j} < b/2 \end{cases}$$

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where

$$S_{b,j} = s_{j-(b-1)/2} + s_{j-(b-3)/2} + \cdots + s_{j+(b-1)/2}.$$

With a rule f, with range r, associate a rule $T_b f$ with range $rb + \frac{1}{2}(b-1)$, i.e. involving a neighbourhood of $(2r+1)b$ sites, defined as follows. Divide the $(2r+1)b$ sites in $2r+1$ blocks of length b. For each block determine the value of the corresponding block variable at time $t-1$. The value at time t of site i given by rule $T_b f$ is, by definition, given by rule f applied to the block variables.

Consider, for example, the range-one CA evolving according to rule 18 of Wolfram (1983). The corresponding function f is such that

$$f(x_1, x_2, x_3) = 1 \qquad \text{iff } (x_1, x_2, x_3) = (0, 0, 1) \text{ or } (1, 0, 0).$$

For $b = 3$ the function $T_3 f$ is such that

$$T_3 f(x_1, x_2, \ldots, x_9) = 1$$

iff

$$x_1 + x_2 + x_3 < \tfrac{3}{2} \qquad x_4 + x_5 + x_6 < \tfrac{3}{2} \qquad x_7 + x_8 + x_9 > \tfrac{3}{2}$$

or

$$x_1 + x_2 + x_3 > \tfrac{3}{2} \qquad x_4 + x_5 + x_6 < \tfrac{3}{2} \qquad x_7 + x_8 + x_9 < \tfrac{3}{2}.$$

A thorough investigation of all legal range-one and totalistic range-two CA shows that the transformation T_b leaves the class unchanged. This result, however, is correct only if the number of sites N of the lattice is large. If this is not the case, i.e. if the ratio b/N is typically greater than a few per cent, then legal class 2, class 3, and class 4 CA behave, after transformation, as class 1 CA.

Figures 1 and 2 represent, respectively, the evolution of typical class 3 and class 4 CA ($r = 2$ totalistic rules 30 and 52 of Wolfram (1984)) for different values of b. The value 0 (respectively 1) is represented by a black (respectively white) square. Initial configurations are disordered, the values 0 and 1 having the same probability. In both

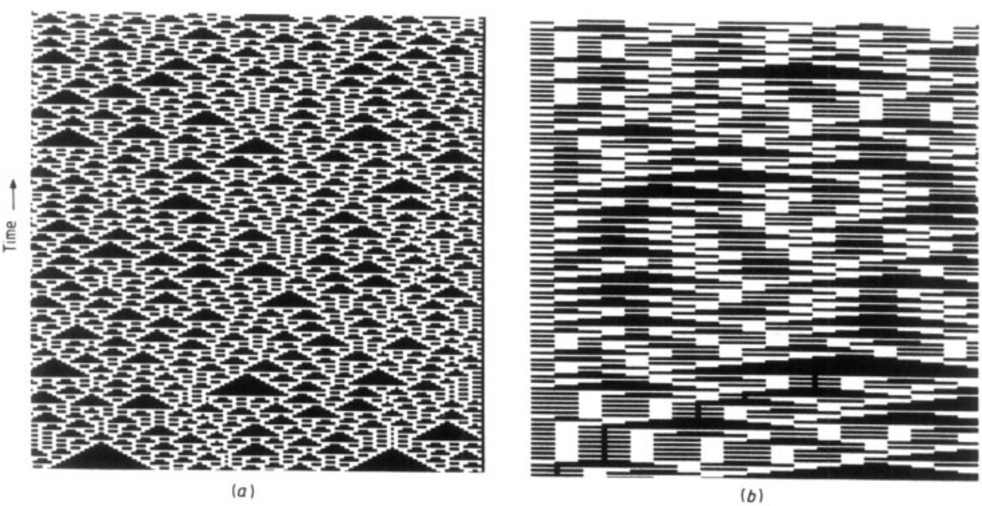

Figure 1. Evolution of class 3 CA with $N = 200\, b$. Only the evolution of the first 200 sites is represented. (a) $k = 2$, $r = 2$ totalistic rule 30, (b) transformed rule for $b = 5$.

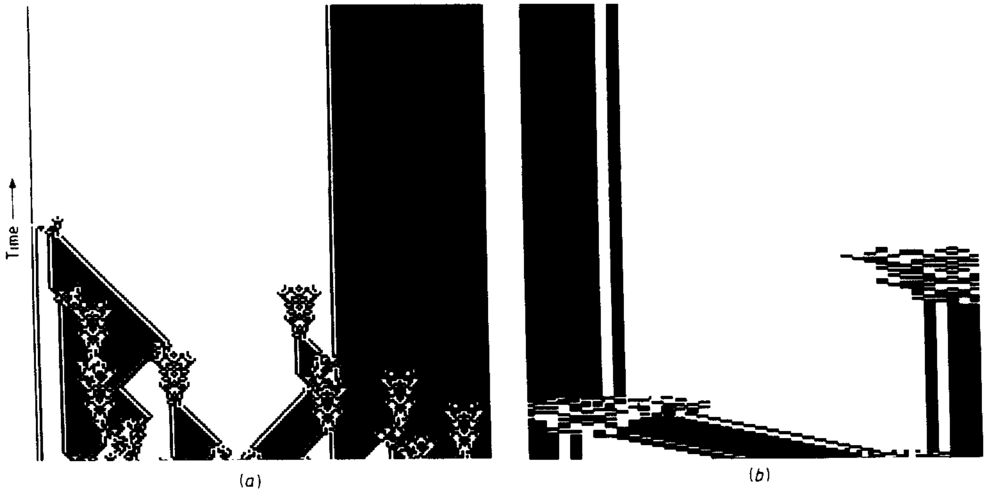

Figure 2. Evolution of a class 4 CA with $N = 200\ b$. Only the evolution of the first 200 sites is represented. (a) $k = 2$, $r = 2$ totalistic rule 52, (b) transformed rule for $b = 5$.

cases, the ratio b/N should not be greater than roughly 3% to leave the respective classes unchanged. The existence, for legal CA, of a 'critical' ratio above which there is a 'transition' to a class 1 CA is due to the increase of the fluctuations with b/N which drives the system into an absorbing state.

The spatio-temporal patterns of figures 1 and 2 for $b > 1$ seem to be stretched in the space direction when compared with the $b = 1$ pattern. After a contraction by a factor b in the space direction (figure 3) the patterns look similar to those obtained for $b = 1$.

In the particular case of class 3 CA, numerous simulations show that the asymptotic density of sites with a non-zero value c is invariant under the transformation T_b. c is often close to $1/k$ (Wolfram 1984). To give clear evidence of the invariance of c under

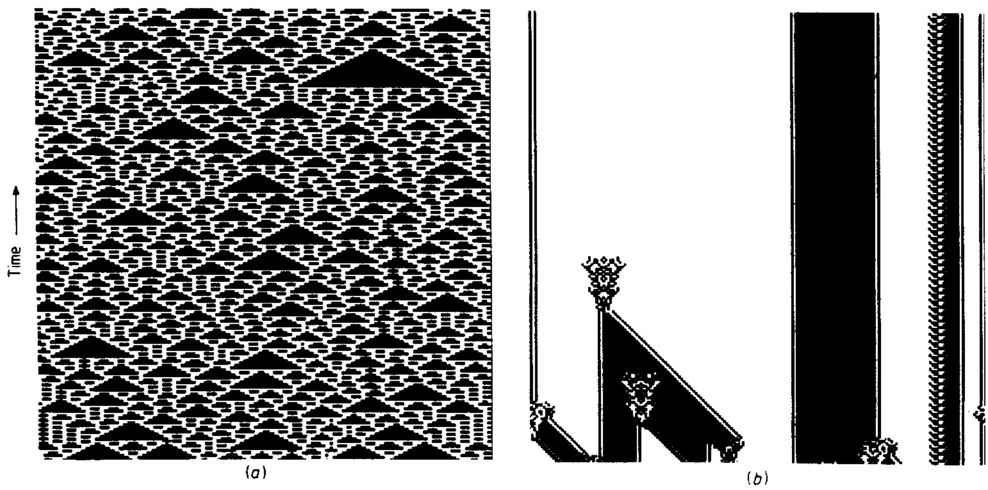

Figure 3. Patterns contracted in the space direction by a factor equal to b with $N = 200\ b$. (a) Rule 30, $b = 5$, (b) rule 52, $b = 5$.

transformation T_b, a CA whose asymptotic density is very different from $\frac{1}{2}$ should be studied, and this is why the $k=2$, $r=1$ CA evolving according to rule 18 has been chosen. Its asymptotic density is exactly equal to $\frac{1}{4}$.

The parameter b characterising the transformed rule $T_b f$ defines a characteristic length and it is not very surprising that quantities like the fluctuations of the asymptotic density which, for $b=1$, scale as $1/N$ have been found to scale as b/N. More precisely, the probability distribution of the asymptotic density c is a function of c and the ratio b/N. Figure 4 illustrates this result; it represents the histogram of c, and the corresponding Gaussian distribution with the same mean c_m and the same variance σ.

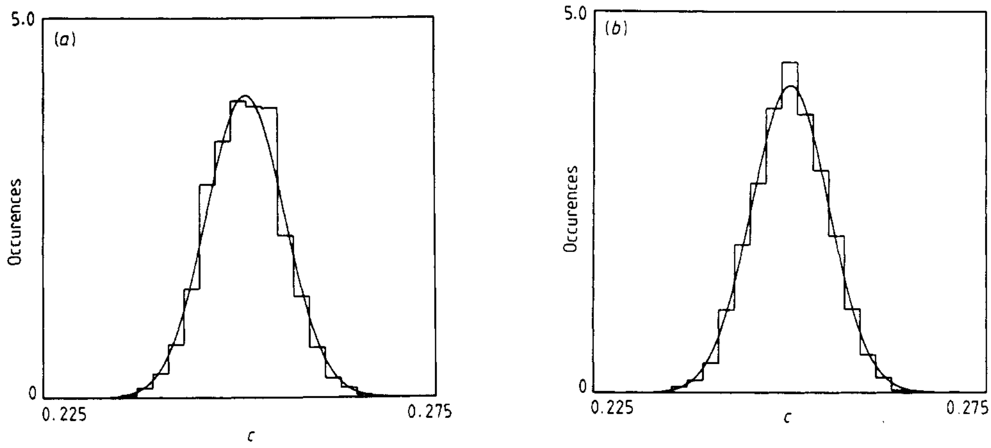

Figure 4. Histogram of the asymptotic density c of (a) the $k=2$, $r=1$ CA rule 18 with $N=5000$, $c_m=0.250\,87$, $\sigma=4.9953\times10^{-3}$; ($b$) the transformed rule for $b=5$ with $N=5000\,b$, $c_m=0.250\,07$, $\sigma=4.9370\times10^{-3}$.

These results suggest that in the $N=\infty$ limit (and b/N small) all the rules $T_b f$ for $b=1,3,5,\ldots$ lead qualitatively and quantitatively to similar evolutions.

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